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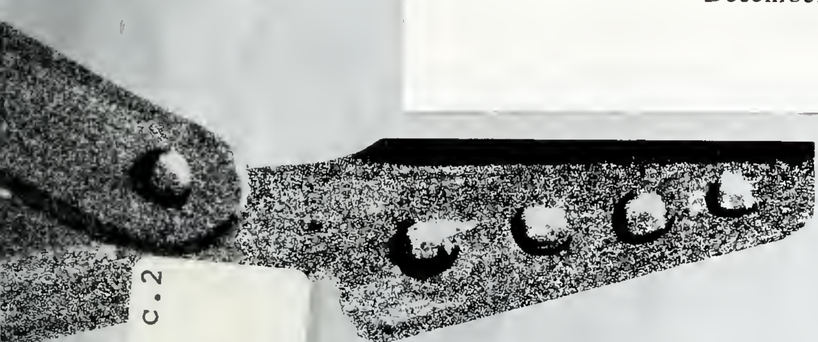

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The Similarity Between Shapes under Affine Transformation

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Abstract

The recognition of similarity between shapes under affine transformation is complicated because we have to consider shapes with rotational, translational, scaling or stretching degrees of freedom. We simplify this problem by introducing the canonical form of a shape. Any shape can be transformed by an affine transformation to a canonical form. The canonical forms of all shapes which are identical under affine transformation can coincide through rotations. A systematical way is provided to find the measure function which can indicate the difference between shapes and keep independent of affine transformation. Then we can reduce the recognition of the similarity between shapes under affine transformation to the computation of the affine-independent difference between their canonical forms, i.e., finding the best matching between a pair of canonical forms by rotation, which is a one dimensional search problem.

We have designed a linear algorithm for transforming a shape to its canonical form. The computation of the difference between a pair of canonical forms can be handled by various current techniques and a new method is presented in this paper. This method is robust in the sense that small changes in the shapes will only cause small changes in the final results. Not only similar shapes can be identified but also the quantity of difference between shapes can be computed.

1. Introduction.

The similarity between shapes is playing an important role in computer vision. Considerable attention has been paid to the problem using the similarity of shape characteristics in object recognition.[2][3][7]]8][9][10] However, there are far fewer discussions on the same problem under an affine transformation.[6] It is common knowledge that an image of an object obtained by human eyes or cameras is actually a projective transformation of the object. When the distance between the viewer and

the object is not very short a projective transformation will be very approximate to an affine transformation. The problem how to recognize whether two shapes are identical or similar under an affine transformation is practically meaningful and difficult.

If we can make two shapes identical by applying an affine transformation to one of them, then we say that these two shapes are identical under affine transformation. To recognize the similarity between shapes under affine transformation, first of all , we have to find some measure which can indicate the "difference" between shapes and is independent of affine transformation, i.e., the difference indicated keeps invariant after the shapes undergo affine transformations. (Most existing measure functions do not have this property). If this kind of difference between shapes is small, then we say the two shapes are similar under affine transformation. The smaller the difference, the more similar the shapes.[1] Moreover, an affine transformation can be decomposed into a translation (two parameters), one rotation (one parameter) and two stretchings along a pair of orthogonal axes (three parameters). Therefore the task of finding the minimum difference between two shapes under affine transformation suffers from a search problem in a six dimensional space.

In this paper we propose an efficient method for recognizing the similarity between shapes under affine transformation. (We consider two dimensional objects). The key point is to define and compute the **canonical form** of an arbitrary shape. Just like the principle in algebra, when we want to know whether two matrices are identical or not we can transform these two matrices to their canonical forms to check if the canonical forms are identical. Using our method any shape can be transformed by an affine transformation to a canonical form, which is centered at the origin point of the coordinates and whose moment along any axis passing through the center equals to 1. If two shapes are identical under affine transformation then their canonical forms can coincide through rotation.

Based on the canonical form of a shape we develop a systematical way to find the functions which can measure the difference between shapes and which are independent of affine transformation.

By introducing the canonical form, the recognition of similarity of shapes under affine transformation can be done by comparing their canonical forms. Then this problem can be reduced to a one dimensional search problem: to find the best match between a pair of canonical forms by rotation.

Because we do not use higher order moments, which are very unsteady, our method is robust in the sense that small changes in the shapes will only cause small changes in the final results obtained.

This paper is organized into 6 sections. In the next section we define dissimilarity functions which measure the "difference" between shapes. In section 3, we discuss the dissimilarity functions which are independent of some transformations (translation, rotation, scaling and affine transformations). We introduce the canonical form of a shape under affine transformation in section 4. In section 5, we give a linear algorithm to compute the canonical form. we discuss how to evaluate the dissimilarity in section 6.

2. Similarity and dissimilarity

We compare shapes by their features. The closeness of features reflects closeness of shapes. To recognize the similarity between shapes we have to define something which can measure the difference between shape features.

Definition 2.1 If for any pair of shapes (P,Q) we have defined a non-negative number $d(P,Q)$ satisfying

(1) $d(P,P)=0$

(2) $d(P,Q)=d(Q,P)$

(3) $d(P,R) \leq d(P,Q)+d(Q,R)$

then we say $d(P,Q)$ is the dissimilarity between P and Q.

According to the definition, two identical shapes are similar; if shape P is similar to Q then shape Q is similar to P; if P is similar to Q and Q is similar to R then P is somehow similar to R. It is the case in practice.

Example 2.1 If we use $area(P)$ to represent the area of P, then $|area(P)-area(Q)|$ is a dissimilarity function.

Example 2.2 Hausdorff distance. First define the distance of a point a from a set P by

$$Hd(a,P)=Min\{|a-b| \mid b \in P\}$$

Then Hausdorff distance is

$$Max\{Hd(a,Q)\,|\,a\in P\}+Max\{Hd(b,P)\,|\,b\in Q\}.$$

Hausdorff distance is a dissimilarity function.

Example 2.3 $area(P+Q)$, where + is the exclusive or operator, is a total dissimilarity function.

Now we discuss some systematical methods for finding dissimilarity functions.

In example 2.1, the dissimilarity function $|area(P)-area(Q)|$ is produced by a mapping $P\rightarrow area(P)\in R$, where R is the set of all real numbers, a distance space. More generally, assume that S is an arbitrary distance space and f is an arbitrary mapping that maps a shape P to a point in S, then we can define a function

$$D_1(P,Q)=d(f(P),f(Q))$$

through the distance function d in the space S.

Theorem 2.1 $D_1(P,Q)$ is a dissimilarity function.

Proof. (1) $D_1(P,P)=d(f(P),f(P))=0$

(2) $D_1(P,Q)=d(f(P),f(Q))=d(f(Q),f(P))=D_1(Q,P)$

(3) $D_1(P,R)=d(f(P),f(R))$

$$\le d(f(P),f(Q))+d(f(Q),f(R))$$

$$=D_1(P,Q)+D_1(Q,R)$$

By this theorem, we can produce a dissimilarity function from any feature of a shape like the perimeter, the diameter, the logarithm of area,..., and so on.

Another general method for obtaining dissimilarity function is as follows.

Theorem 2.2 If $d(P,Q)$ is a dissimilarity function and f is a concave non-decreasing function satisfying that $f(0)=0$, then $D_2(P,Q)=f(d(P,Q))$ is a dissimilarity function.

Proof. (1) $D_2(P,P)=f(d(P,P))=f(0)=0$

(2) $D_2(P,Q)=f(d(P,Q))=f(d(Q,P))=D_2(Q,P)$

(3) $D_2(P,R)=f(d(P,R))$

$\le f(d(P,Q)+d(Q,R))$ (because f is non-decreasing)

$$=f(d(P,Q)+d(Q,P))+f(0)$$

$$\leq f(d(P,Q))+f(d(Q,R)) \text{ (because } f \text{ is concave)}$$

$$=D_2(P,Q)+D_2(Q,R)$$

We can have a sequence of "weaker" and "stronger" dissimilarity functions between pairs of shapes, the stronger requiring that more features of the shape be similar before they are close. By introducing a dissimilarity combination function we can take more features of a shape into account, and make the dissimilarity function stronger.

Definition 2.4 A dissimilarity combination function $c(d_1,d_2)$ is a function satisfying that if $d_1(P,Q)$, $d_2(P,Q)$ are dissimilarity functions then so is $c(d_1(P,Q),d_2(P,Q))$.

Example 2.4 For any $p \geq 1, c(d_1,d_2)=(\lambda_1 d_1^p+\lambda_2 d_2^p)^{1/p}$ is a dissimilarity combination function, where λ_1 and λ_2 are used to weight $d_1(P,Q)$, $d_2(P,Q)$, respectively.

3. Invariant Dissimilarity and Independent Dissimilarity Functions.

The similarity between shapes should be at least independent of translation, scaling and rotation. Therefore, a good dissimilarity function should reflect this property.

Definition 3.1. Let G be a transformation group on a plane. A dissimilarity function $d(P,Q)$ is G invariant if

$$d(\tau P, \tau Q)=d(P,Q) \text{ for all } \tau \in G.$$

A dissimilarity function $d(P,Q)$ is G independent if

$$d(\tau' P, \tau'' Q)=d(P,Q) \text{ for all } \tau', \tau'' \in G.$$

The examples of dissimilarity functions in the last section are all translation, rotation invariant, but not scaling invariant. Both Hausdorff distance and function $area(P+Q)$ are neither translation independent nor rotation independent. All of them are not scaling independent.

As mentioned earlier, we are interested in recognizing the shape similarity under affine transformation. Under an affine transformation the shape of an object can undergo an arbitrary scaling, translation, rotation and stretching (a compression

is regarded as a stretching with factor less than 1) along a pair of orthogonal axes. The situation is more complicated. All examples in the last section are neither affine independent nor affine invariant. It seems not easy to find affine invariant dissimilarity functions. It is even harder to find affine independent dissimilarity ones.

Here we give an example of affine invariant dissimilarity function

Example 3.1 $d(P,Q)=area(P+Q)/area(P\cup Q)$.

It is not trivial to prove that this function qualifies for an affine invariant dissimilarity function. Notice that this example is not affine independent.

we can prove that if d is a G invariant dissimilarity function, then

$$D_3(P,Q)=\underset{\tau\in G}{Min}\{d(P,\tau Q)\}$$

is a G independent dissimilarity function. However, because $\underset{\tau\in G}{Min}\{d(P,\tau Q)\}$ is difficult to compute, this method is impractical. In the next section we will present a better approach.

4. Canonical Form under Affine Transformation

Before discussing the canonical form of a shape under affine transformation, let us introduce the moment curve of a shape first.

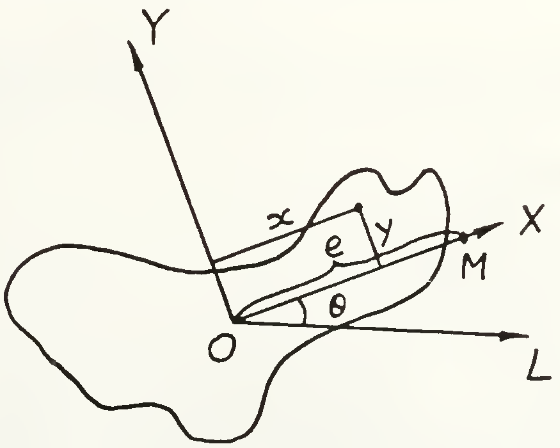

Fig. 1

As shown in Fig. 1, point O is the center of gravity of a shape P, and L is a line passing through O. Choose coordinate axes such that the origin point O coincides

with the center of gravity of the shape P and X-axis coincides with line L. The moment of P is the integration

$$e = \int_P y^2 dxdy.$$

We take a point M on OX such that its distance from the origin point O equals to this integration. We rotate the coordinate axes about the origin point O and denote the angle between line L and the X-axis by θ. Then θ changes from 0 to 2π. By tracing point M on X-axis, we obtain a curve. By a well known conclusion from geometry, the curve can be expressed as

$$e = a + b\,Cos(2\theta+\phi),\ 0 \le \theta \le 2\pi,\ a \ge b \ge 0.$$

This curve is called the moment curve.

Now we introduce a very useful concept: the canonical form of shapes under affine transformation.

Definition 4.1 A shape is in canonical form, if it satisfies that

(1) The center of gravity is at the origin point.

(2) Its moment curve is a unit circle.

Theorem 4.1 Any shape with positive area can be transformed to a canonical form by an affine transformation.

Proof. The moment curve can be expressed as $e = a + b\,cos(2\theta+\phi)$, $0 \le \theta \le 2\pi$, $a \ge b \ge 0$. When $\theta = -\phi/2$ the value of e will reach its maximum, while when $\theta = \pi/2 - \phi/2$ the value of e will reach its minimum. These two angles are orthogonal. By two stretches along two orthogonal axes corresponding to these two angles, we can transform P into another shape P' such that the moments of P' along these two directions ($\theta = -\phi/2$ and $\theta = \pi/2 - \phi/2$) are equal to 1. Let the moment curve of P' be expressed by $e = A + B\,Cos(2\theta+\phi)$. Then we have

$$1 = A + BCos(-\phi+\phi) = A + B.$$

$$1 = A + BCos(\pi-\phi+\phi) = A - B.$$

Therefore A=1 and B=0. The equation becomes

$$e = 1.$$

This means the moment curve of P' is a unit circle. To transform a shape to its canonical form we need to translate the shape such that its center of gravity

coincides with the origin point, and stretch the shape along a special pair of orthogonal axes. therefore, any shape can be transformed to its canonical form by an affine transformation.

Theorem 4.2 If P and Q are identical under affine transformation and $C(P)$,$C(Q)$ are their canonical forms, then $C(P)$ and $C(Q)$ can coincide through a rotation about the origin point, i.e., for any affine transformation τ there is a rotation r about the origin point such that $C(\tau P)=rC(P)$.

Proof. Affine transformation form a group. If τ is an affine transformation then so is τ^{-1}; if τ_1 and τ_2 are affine transformations then so is $\tau_1\tau_2$. $C(P)$ can be transformed to P, P can be transformed to Q and Q can be transformed to $C(Q)$, by affine transformations. Therefore $C(P)$ can be transformed to $C(Q)$ by an affine transformation τ, i.e., $C(P)$ can coincide with $C(Q)$ through τ. Any transformation can be decomposed into a translation, two stretches and a rotation, therefore we can write $\tau=\tau_0\kappa_1\kappa_2 r$ where τ_0 is a translation, κ_1,κ_2 are two stretches along two mutually orthogonal axes passing through the origin point O and r is a rotation centered at the origin point. Since $C(P)$ and $C(Q)$ have the same center of gravity (the origin), the translation τ_0 must be identical, $\tau_0=e$, where e is the identity transformation. Because, if $\tau_0=!e$, the τ_0 would translate the origin point O to a non-origin. But κ_1,κ_2,r can only transform origin to origin, and non-origin to non-origin, therefore $\tau=\tau_0\kappa_1\kappa_2 r$ would transform the origin point to a non-origin, a contradiction. Since $C(P)$ and $C(Q)$ have the same moment curve (the unit circle), we must have $\kappa_1=\kappa_2=e$. Because if any of them were not the identical transformation e, then they would change the moment curve from a unit circle to some thing else, which is impossible. Therefore $\tau=eeer=r$ and $C(P)$ can be transformed by a rotation $\tau=r$ to $C(Q)$.

In the last section we mentioned that it is not easy to find an affine invariant or independent dissimilarity function. But now based on the canonical form, we can find many of them.

Theorem 4.3 If $d(P,Q)$ is a rotation invariant dissimilarity, then

$$D_4(P,Q)=\underset{r}{MIN}\{d(C(P),rC(Q))\}$$

is an affine invariant dissimilarity. The minimum is among all rotations.

Proof. It is a dissimilarity function because:

(1) $D_4(PP)=\underset{r}{Min}\{d(C(P),rC(P))\}\le d(C(P),eC(P))=0.$

(2) $D_4(P,Q)=\underset{r}{Min}\{d(C(P),rC(Q))\}$

$=\underset{r}{Min}\{d(r^{-1}C(P),r^{-1}rC(Q))\}$ (d is rotation invariant)

$=\underset{r}{Min}\{d(r^{-1}C(P),C(Q))\}$ $(r^{-1}r=e)$

$=\underset{r}{Min}\{d(C(Q),r^{-1}C(P))\}$ (d is symmetric)

$=\underset{r}{Min}\{d(C(Q),rC(P))\}$ (the set $\{r^{-1}\}$ is the same as $\{r\}$)

$=D_4(Q,P).$

(3) $D_4(P,R)=\underset{r}{Min}\{d(C(P),rC(R))\}$

$=\underset{r}{Min}\{d(r^{-1}C(P),r^{-1}rC(Q))\}$ (d is rotation invariant)

$=\underset{r}{Min}\{d(r^{-1}C(P),C(Q))\}$ $(r^{-1}r=e)$

$=\underset{r',r''}{Min}\{d(C(P),r'r''C(R))\}$ (the set $\{r'r''\}$ is the same as $\{r\}$)

$\le\underset{r',r''}{Min}\{d(C(P),r'C(Q))+d(r'C(Q),r'r''C(R))\}$ (triangular inequality)

$=\underset{r',r''}{Min}\{d(C(P),r'C(Q))+d(C(Q),r''C(R))\}$ (d is rotation invariant)

$=\underset{r'}{Min}\{d(C(P),r'C(Q))\}+\underset{r''}{Min}\{d(C(Q),r''C(R))\}$

$=D_4(P,Q)+D_4(Q,R).$

It is affine invariant, because:

$D_4(\tau'P,\tau''Q)=\underset{r}{Min}\{d(C(\tau'P),rC(\tau''Q))\}$ (definition)

$=\underset{r}{Min}\{d(r'C(P),rr''C(Q))\}$ (Theorem 4.2)

$=\underset{r}{Min}\{d(r'C(P),r'rC(Q))\}$ (the set $\{r\}r''$ is the same as $r'\{r\}$)

$=\underset{r}{Min}\{d(C(P),rC(Q))\}$ (d is rotation invariant)

$=D_4(P,Q).$

All examples of dissimilarity functions presented in this paper are rotation invariant. From each of them we can obtain an affine independent dissimilarity function.

Another general way to get affine invariant dissimilarity functions is as follows.

Theorem 4.4 Assume that S is a distance space with distance function d, and that f is a rotation invariant mapping which maps the canonical form $C(P)$ to $f(C(P)) \in S$. Then

$$D_5(P,Q)=d(f(C(P),f(C(Q)))$$

is an affine invariant dissimilarity function.

Proof. It is easy to show D_5 is a dissimilarity function. It is affine invariant, because:

$$D_5(\tau'P,\tau''Q)=d(f(C(\tau'P)),f(C(\tau''Q))) \quad \text{(definition)}$$

$$=d(f(r'C(P)),f(r''C(Q))) \quad \text{(Theorem 4.2)}$$

$$=d(f(C(P)),f(C(Q))) \quad (f \text{ is rotation invariant})$$

$$=D_5(P,Q).$$

Theorem 4.5 If d_1 and d_2 are affine invariant dissimilarity functions, then so is any combination $C(d_1,d_2)$.

Proof. Obvious.

By introducing the canonical form we developed a systematical way to find affine independent dissimilarity functions. In fact, we can prove that any affine independent dissimilarity function can be defined this way. This is the first reason why we discuss the affine canonical form. The second reason is that using the canonical form, the computation of the dissimilarity between two shapes under affine transformation can be considerably simplified, as shown in the next two sections.

5. Find the Canonical Form

In this section, we show how to compute the canonical form.

The first step in finding the canonical form is to compute the center of gravity and translate the shape to make the center of gravity coincided with the origin.

Now it is necessary to compute the parameters (the angle of axis and two factors) of the stretches that will change the moment curve to a unit circle.

The moment curve is expressed by equation

$$m = a + bCos(2\theta + \phi).$$

Let us evaluate the moments at $\theta = 0, 2\pi/3, 4\pi/3$, (see Fig. 1) and obtain m_1, m_2, m_3. We have

$$m_1 = a + bCos\phi$$

$$m_2 = a + bCos(2\pi/3 + \phi)$$

$$m_3 = a + bCos(4\pi/3 + \phi)$$

Adding them together, we obtain

$$m_1 + m_2 + m_3 = 3a,$$

$$a = \frac{1}{3}(m_1 + m_2 + m_3).$$

Squaring them,

$$(m_1 - a)^2 = b^2 Cos^2\phi = \frac{b^2}{2} + \frac{1}{2}Cos2\phi$$

$$(m_2 - a)^2 = b^2 Cos^2(2\frac{\pi}{3} + \phi) = \frac{b^2}{2} + \frac{1}{2}Cos(4\frac{\pi}{3} + 2\phi)$$

$$(m_3 - a)^2 = b^2 Cos^2(4\frac{\pi}{3} + \phi) = \frac{b^2}{2} + \frac{1}{2}Cos(2\frac{\pi}{3} + 2\phi)$$

Adding them together we obtain

$$(m_1 - a)^2 + (m_2 - a)^2 + (m_3 - a)^2 = \frac{3}{2}b^2.$$

$$b = \sqrt{\frac{2}{3}((m_1 - a)^2 + (m_2 - a)^2 + (m_3 - a)^2)}.$$

$$\phi = \mathrm{arc}\,Cos\frac{m_1 - a}{b}.$$

When $\theta = -\phi/2$, m arrives its maximum value a+b. When $\theta = \pi/2 - \phi/2$, m arrives its minimum value $a - b$. Therefore after stretching by $\sqrt{a+b}$ along

direction $\theta=-\frac{\phi}{2}$ and by $\sqrt{a-b}$ along direction $\theta=\pi/2-\phi/2$, the shape should be in a canonical form.

It is easy to see that the transformation of a shape to its canonical form can be finished in $O(n)$ time, where n is the total number of points or line segments in the representation of the shape.

6. Computing the Dissimilarity between Shapes

Having changed two shapes into their canonical forms, the recognition of the similarity under affine transformation becomes easier. If two shapes are identical under affine transformation, their canonical forms will coincide through a rotation. If we want to detect the closeness of the shapes under affine transformation, we can choose some affine independent dissimilarity functions to measure the difference.

We have already pointed out that the most general affine invariant dissimilarity function is $D_4(P,Q)=\underset{r}{Min}\{d(C(P),rC(Q))\}$,which is induced from a rotation invariant dissimilarity function $d(P,Q)$. To compute the value of $D_4(P,Q)$, we should find the rotation r that transforms C(Q) to a best match position with C(P). This is a minimizing problem in one dimensional space.

Many methods including various heuristic algorithms can be used for the problem of best match in one dimensional space. In this section, we propose a new one. Let

$$\Phi(r)=d(C(P),rC(Q))$$

The function $\Phi(r)$ usually satisfies Lipsiz condition:

$$|\Phi(r_1)-\Phi(r_2)|\leq L\,|r_1-r_2|.$$

with a coefficient L depending on the dissimilarity function.

Given an error tolerance ϵ, a straightforward way to compute $d(C(P),rC(Q))$ is to try all values of r with a step length ϵ/L. Then we need $\frac{L}{\epsilon}$ steps.

In fact, we can make use of the property $|\Phi(r_1)-\Phi(r_2)|\leq L\,|r_1-r_2|$ still further. As illustrated in Fig. 2, we draw line S of slope $-L$ through point $(a,\Phi(a))$ and a line T through point $(b,\Phi(b))$. These two lines intersect at point (u,v). The Lipsiz condition guarantees that if $a\leq x\leq b$ then $\Phi(x)\geq v$. It is easy to show that $v=(\Phi(a)+\Phi(b))/2-L(b-a)/2$, therefore

$$\Phi(x) \geq (\Phi(a)+\Phi(b))/2 - L(b-a)/2$$

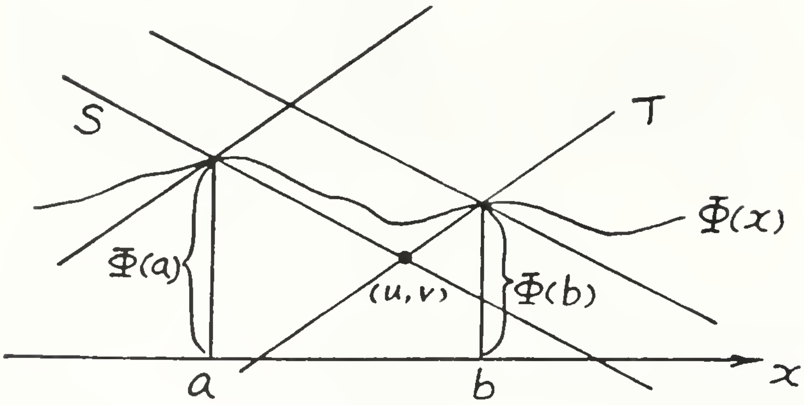

Fig. 2

Based on this fact, we will try to skip as many computations as we can.

Now we divide our computation into phases. At phase i, we compute $\Phi(r)$ for all values of $2k_i\pi$, $k_i = 1/2^i, 3/2^i, 5/2^i, \ldots, (2^i-1)/2^i$.

In other words,

Phase 0: $k_0 = 1$;

Phase 1: $k_1 = 1/2$;

Phase 2: $k_2 = 1/4, 3/4$;

Phase 3: $k_3 = 1/8, 3/8, 5/8, 7/8$;

Assume that we have finished phase i and m is the minimum value of Φ that we have ever found. By the reason mentioned above, we can easily see that if an interval $[j/2^i, (j+1)/2^i]$ has the property that

$$(\Phi(j/2^i) + \Phi((j+1)/2^i))/2 - L/2^{i+1} \geq m$$

Then we can guarantee that no points in this internal can have a lower value than m, therefore we can remove all these kinds of intervals.

At phase $i+1$, of course, we need not try points inside these intervals. The procedure goes on until $1/2^i < \epsilon / L$.

This method can be improved upon by incorporating heuristics. At the very beginning we use heuristics to find some promising initial position match, i.e., make the initial value of $\Phi(r)$ small. This will lead to skipping many point tests. For example, when the best matching happens, the farthest point of $C(P)$ from the center of gravity will match the forthest point of $C(Q)$. Therefore we should first rotate $C(P)$ and $C(Q)$ so that the farthest points coincide, then start the test.

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